

# Optimized-Cost Repair in Multi-hop Distributed Storage Systems with Network Coding

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**Abstract**—In distributed storage systems reliability is achieved through redundancy stored at different nodes in the network. Then a data collector can reconstruct source information even though some nodes fail. To maintain reliability, an autonomous and efficient protocol should be used to repair the failed node. The repair process causes traffic and consequently transmission cost in the network. Recent results found the optimal traffic-storage tradeoff, and proposed regenerating codes to achieve the optimality. We aim at minimizing the transmission cost in the repair process. We consider the network topology in the repair, and accordingly modify information flow graphs. Then we analyze the cut requirement and based on the results, we formulate the minimum-cost as a linear programming problem for linear costs. We show that the solution of the linear problem establishes a fundamental lower bound of the repair-cost. We also show that this bound is achievable for minimum storage regenerating, which uses the optimal-cost minimum-storage regenerating (OCMSR) code. We propose surviving node cooperation which can efficiently reduce the repair cost. Further, the field size for the construction of OCMSR codes is discussed. We show the gain of optimal-cost repair in tandem, star, grid and fully connected networks.

**Index Terms**—Distributed storage systems, regenerating codes, minimum-cost, surviving node cooperation, linear optimization.

## I. INTRODUCTION

THE increasing number of new applications such as voice and video over Internet, video on demands, user generated content (like YouTube service) have caused a fast growth of traffic over the wired and wireless networks during the last decades. This traffic growth requires big data storages. Consequently, the idea of distributing files among some storage nodes on the network has been proposed. High demand in QoS (quality of service) in e.g., fast and ubiquitous access, low-delay and reliability poses high requirements on distributed information storage. Compared to centralized storage systems, the benefits of distributed storage systems include fast and ubiquitous access, high reliability, availability and scalability. For the benefits, recently distributed storage systems have been used in big data centers like Google file systems [1], Oceanstore [2] and Total Recall [3].

On another aspect, the distributed storage system has brought the higher reliability of the measured data in wireless sensor networks [4]. Wireless sensor networks consist of several small devices (sensors) which measure or detect a

physical quantity e.g., temperature, pressure, light and so on. The main characteristic of the sensors are limited battery, low CPU power and limited communication capability [7]. Because of the limited communication capability of sensors, communication between nodes is generally through multi-hop. That is, storage nodes may relay the message of other nodes.

The storage nodes are vulnerable due to disk failure, power off, or a node leaving the system. Thus to make the data reliable over the unreliable nodes, the data is encoded and distributed among storage devices [5], [6], [7]. Reliability is usually obtained through redundant nodes in a distributed storage system, in which error control (EC) codes are normally used to increase the storage efficiency. An EC code with the *MDS (maximum distance separable)* property is optimal in term of the redundancy-storage tradeoff. If a file of size  $M$  bits is coded by  $(n, k)$ -MDS codes ( $k \leq n$ ) and distributed among  $n$  nodes, each node stores  $M/k$  bits. Then every  $k$  nodes can reconstruct the original file. Nonetheless, if a node fails, and a new node joins the system, the new node needs to download  $M$  bits to regenerate  $M/k$  bits. Thus, it may not be optimal considering the traffic for regenerating a new node. In [5], [6], the authors investigate the problem of the repairing traffic (bandwidth) and find the optimal storage-bandwidth tradeoff. A new class of erasure codes, namely regenerating codes based on network coding ([8], [9]), are proposed in [5], [6] to achieve the optimal tradeoff. In the repair process, the new node may not have the same encoded data as the failed node. However, the restored data has the same MDS property (i. e., with the new node still every  $k$  nodes can reconstruct the original file). We call this property as *regenerating code property (RCP)*. This kind of repair is generally called functional repair, in contrast to the exact repair, where the new node stores the same data as the failed node. The exact regenerating of a new node is studied in [10], [11], [12].

Two extreme scenarios in the fundamental storage-bandwidth tradeoff of regenerating codes [5] correspond to two types of regenerating codes: one scenario, namely the minimum storage regenerating (MSR) code, uses the same amount storage space as the MDS code but with lower repair traffic. Another scenario, namely the minimum bandwidth regenerating (MBR) code has slightly more storage per node, but its repair bandwidth is considerably lower than the MSR

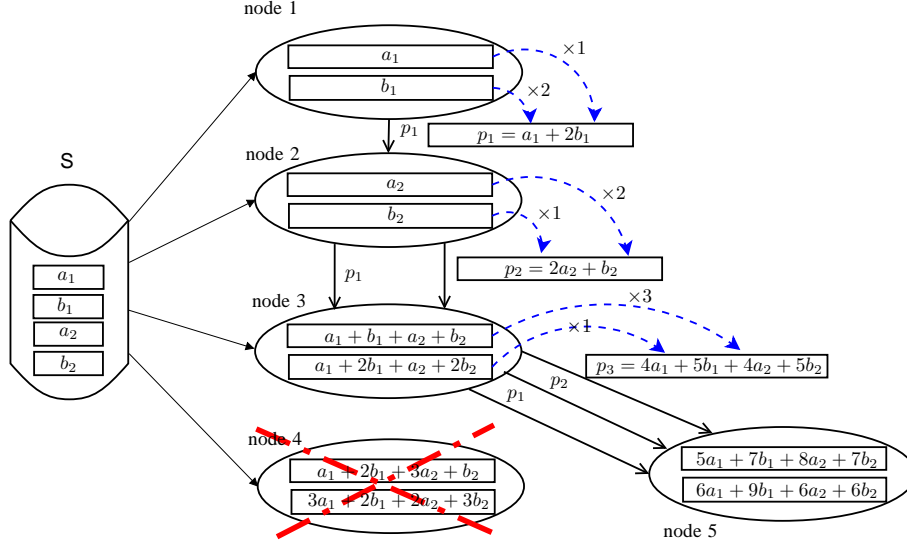


Fig. 1. A distributed storage system in a 4-node tandem network. Dashed lines denote how network codewords are formed. Node 4 fails and node 5 is the new node. For regenerating a new node,  $p_1, p_2, p_3$  are formed by linear combination of fragments in node 1, 2, and 3, respectively. The underlying finite field is  $\text{GF}(11)$ .

(MDS) codes [5]. To further decrease the repair traffic in the scenario of multiple simultaneously-failed nodes, reference [13] proposes new node cooperative repair. In new node cooperative regenerating codes, the new nodes cooperate to minimize the repair bandwidth. The difference between new node cooperative regenerating and our method is that in our approach surviving nodes cooperate, i.e., surviving nodes can combine the received data with their own information. These two approaches together can provide a fully cooperative regenerating process.

Though references [5], [6], [13], [10] have well addressed the optimal storage-bandwidth tradeoff in distributed systems from different aspects, the link cost (transmission cost in channels) and the impact of the network topology have not been considered. In a practical system, the cost is an important design consideration and different links (channels) of a network may have different costs. The topology of networks may also impact the repair cost. Looking through two main applications for distributed storage systems in data centers and wireless sensor network, we find that data centers have hierarchical network structures [17] and wireless sensor networks have multi-hop network structures [21]. Recently, reference [14] considers the transmission cost from surviving nodes to the new node and the cost-bandwidth tradeoff is derived. This model suits for the scenarios in which there are direct links from surviving nodes to the new node. Yet, this model does not exploit the network structure. For instance, consider two storage nodes that have direct links to the new node as well as a link between themselves. Suppose these surviving nodes want to send data to the new nodes. Instead of direct communication, the surviving nodes could cooperate and send an aggregated data to the new node. Thus, we propose the idea of surviving nodes cooperation in order to reduce the transmission cost. This problem is also interesting in distributed storage systems in wireless sensor networks

(WSN), where reducing transmission cost is demanding. In [30] it has been shown that the optimal-cost repair can be found also in a decentralized approach which suits for WSN due to the lack of a central node and CPU power limitation of nodes.

Other related work includes followss. Reference [15] proposes a tree-structure algorithm for minimum storage regenerating (MSR) codes. The work considers a network having links with different bandwidth (capacity). Then by exploiting Prim's algorithm, they find a repair approach which minimizes the repair time (i.e., finding routes in network with the highest capacity to maximize the speed of repair process). Our work is distinct from [15] since we minimize the repair-cost instead of repair-time. Repair-cost can be formulated as the summation of costs on all links in network, while repair-time depends on the bottleneck of a route.

We shall study the repair-cost in multi-hop networks where links may have different transmission costs. To formulate the problem, we modify the information flow graph by taking into considering the network topology. Then, by cut-set bound analysis and solving a linear programming problem, we derive a lower bound of repair costs. In distributed storage systems, networks evolve along with time. Thus, there is infinite stages of repair. In general, one should run cut-set bound analysis in infinite stages to find the optimum cost. We first give a lower bound of repair-cost by cut-set analysis in only first stage of repair. Later, we prove that this lower bound is tight for minimum storage regenerating codes which are the optimum-cost minimum storage regenerating codes. To the best of our knowledge, we are the first who consider *surviving node cooperation (SNC)* scheme to reduce the repair cost. SNC allows intermediate nodes in the network to combine their received fragments from other surviving nodes and their own stored data. For a general setting, we formulate the minimum-cost problem and study the solution. Related work includes

[23], in which the minimum-cost multicast problem is studied in networks with network coding. Reference [24] studies the transmission cost of distributing a source file among nodes on the network with or without network coding. It is shown in [24] that by file splitting the optimality can be achieved. Here we assume a source file is already distributed among different nodes. Our objective is to find the optimum repair process when a storage node fails, i.e., how to regenerate a new node with the minimal cost. It may worth to note that our approach holds for every network topology. The algorithm takes the cost of transmitting one unit of information between each pair of nodes as its input and find the optimum cost repair. In addition, we try to find a closed form for the repair cost for some specific topologies like tandem, star, grid, and fully connected networks.

The remainder of the paper is organized as follows. In Section II, we will give a motivation example by a specific network. For more general networks, we formulate optimization problem in Section III. In Section IV, we discuss the optimal-cost repair when the storage capacity per node is  $\alpha = M/k$ . In Section V we investigate the fundamental lower bound of repair costs in tandem, star, grid, and fully connected networks and present the gain of the optimal-cost repair in those networks. Finally, in Section VI, we conclude the paper.

TABLE I  
NOTATIONS

Symbol	Definition
$n$	number of storage nodes in a distributed storage system
$k$	minimum number of nodes needed to-reconstruct the original file
$d$	number of surviving nodes for a repair process
$\alpha$	storage amount of individual node
$\beta$	amount of data downloads from each surviving node in the repair process
$M$	original information file size
$\underline{s}$	$M \times 1$ vector denoting the original file
$\underline{Q}_i$	$M \times \alpha$ matrix denoting coding coefficients of node $i$
$\underline{X}_i$	$\alpha \times 1$ vector denoting the content of node $i$ ,
i.e., $\underline{X}_i = \underline{Q}_i^T \underline{s}$	
$\sigma_{non-opt}$	repair cost by non-optimized-cost approach (repair-cost without SNC)
$\sigma_c$	repair cost by optimized approach (repair-cost with SNC and optimization)
$g_c$	optimization gain, $g_c = \sigma_{non-opt}/\sigma_c$

## II. MOTIVATING EXAMPLE

We first give an example to illustrate the motivation. Consider a distributed storage system in a four-node tandem network shown in Fig. 1. Here we follow the notation of the regenerating codes [5], [6]. The notations are also given in Table I. We divide a source file into equal size fragments (packets). A file of size 4 mega-bits ( $M = 4$  fragments; One fragment has 1 mega-bit) is coded with a regenerating code [5] and distributed among 4 nodes ( $n = 4$ ). Each node stores 2 one-mega-bit fragments ( $\alpha = 2$ ) and the source file can be reconstructed by any 2 nodes ( $k = 2$ ). When a node fails (say node 4), a new node downloads  $\beta$  fragments from each of 3 surviving nodes ( $d = 3$ ).

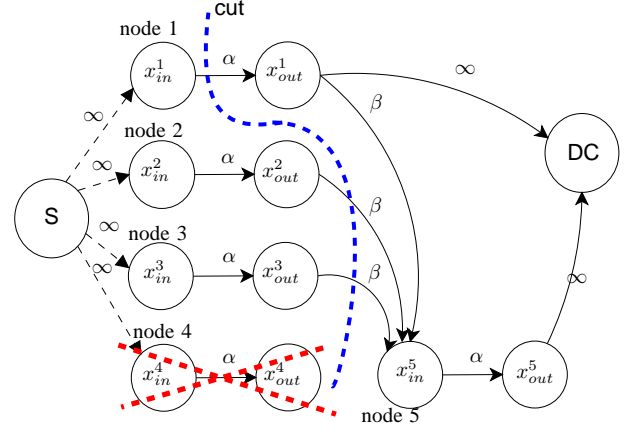


Fig. 2. Cut-set bound analysis in the information flow graph. Node 4 fails and node 5 is the new node. There exist direct links from surviving nodes to the new node.

If the new node has direct links to all the surviving nodes, the distributed storage system can be represented by a directed acyclic graph with a failure/repair on node 4, as shown in Fig. 2. The graph is known as an information flow graph [5]. In the graph, there is a source node ( $S$ ) connected to storage nodes through infinite-capacity links. Each storage node is depicted by input ( $in$ ) and output ( $out$ ) nodes connecting by a  $\alpha$ -capacity link. When a node fails, a new node downloads  $\beta$  fragments from each node. A  $DC$  denoting a data collector reconstructs the source file from any  $k$  storage nodes.

By cut-set bound analysis on the information flow graph, it is shown that there is the fundamental storage-bandwidth tradeoff to regenerate a new node [5]. A cut in the flow graph refers to a set of edges in which network nodes are divided into two separated parts; one part contains  $S$  and another part contains the  $DC$ . The value of a cut is the sum of capacities of edges from the source to the destination direction. Then, the corresponding cut must meet a fundamental requirement resulting from min-cut max-flow theorem for multicast networks [8]. The theorem expresses that in a multicast network destinations can reconstruct the source file whenever all the cuts between source and destinations are greater than or equal to the source file size.

By the cut-set bound analysis of the information flow graph in Fig. 2, we can see that for a  $DC$  to reconstruct the source file, it requires  $\alpha + 2\beta \geq M$ . For  $\alpha = 2$ ,  $M = 4$ , then  $\beta \geq 1$ . That is, the new node must download at least  $\beta = 1$  mega-bits from each surviving node. The optimal-traffic repair to achieve this lower bound is to download  $p_1, p_2, p_3$  as in Fig. 1 from node 1, node 2 and node 3, respectively. Here  $p_1, p_2$  and  $p_3$  are formed by linear coding at nodes 1, 2, and 3, respectively. Yet, the analysis is different if we consider the link cost.

We assume that each fragment of 1 mega-bits transmitted in a single channel costs one transmission unit. To reach the new node (node 5), we can easily see that  $p_1$  passes the route (node 1  $\mapsto$  node 2  $\mapsto$  node 3  $\mapsto$  node 5) with a cost of 3 units, and  $p_2$  passes node 2  $\mapsto$  node 3  $\mapsto$  node 5 with a cost of 2 units, and  $p_3$  passes node 3  $\mapsto$  node 5 with a cost of 1 unit. Thus, the total cost in the repair is 6 units. In this

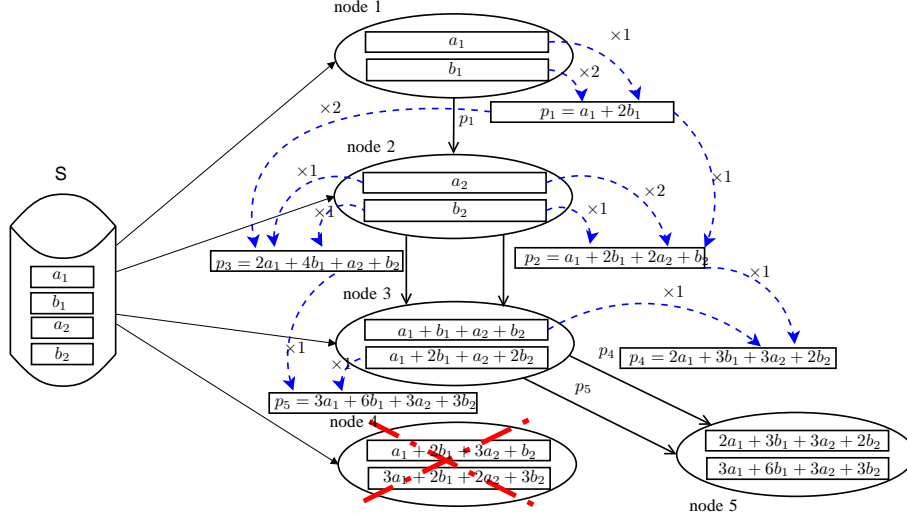


Fig. 3. Regenerating by surviving node cooperation in a tandem network. Here, for regenerating a new node, fragment  $p_1$  is formed by combining fragments of node 1. Fragments  $p_2, p_3$  are formed by linear combination of fragment  $p_1$  and stored fragments in node 2. Finally, fragments  $p_4, p_5$  are formed by linear combination of  $p_2, p_3$ , received fragments on node 3, with the stored fragments on node 3.

paper, we call *non-optimized-cost repair* to the approach which minimizes the repair-bandwidth (without optimizing the cost) and the corresponding cost is denoted as  $\sigma_{non-opt}$ .

Non-optimized-cost repair may be optimal in term of repair bandwidth. Yet it might not be optimal in terms of the repair cost. For example, consider the repair scheme in Fig. 3, where we allow surviving node cooperation (SNC). Here the cooperation means that one surviving node can combine/encode the coded symbols of another node. That is, surviving nodes are allowed to linearly combine their own fragments with the received fragments from other nodes. For instance, at node 2,  $p_2$  and  $p_3$  are encoded from the received  $p_1$  from node 1 and stored fragments in node 2. At node 3,  $p_4$  and  $p_5$  are obtained by encoding  $p_2$  and  $p_3$  with the fragments of node 3. Note that here we only consider functional repair, in which the regenerated node may not be identical to the failed node but it has the same code property. That is, with the new node (node 5), any 2 out of 4 nodes can reconstruct the source. It is easy to see that the cost of repair is reduced to 5 units as shown in Fig. 3 (only two fragments are transmitted from node 3 to node 5). The example shows SNC can reduce the transmission cost. We note that the repair approach in Fig. 3 still is not optimal in term of the transmission cost. We shall show in the next section that the optimal repair cost in this example is 4. We further note that SNC can be applied to the scenario of one or multiple node failure, but the cooperative regenerating codes in [13] and [22] can only be used for the scenario of multiple node failure.

### III. PROBLEM FORMULATION

In this section, we first introduce the cost matrix which presents the cost of transmitting one unit of data (e.g., one fragment) between any pair of helper nodes. In our approach surviving nodes cooperate in the repair process. Thus, we assume storage nodes are capable of performing linear calculation in finite fields. The number of helper nodes, which

is denoted as  $d$ , is assumed to be greater than  $k$ , i.e.,  $d \geq k$  (for  $d < k$  successful regenerating is not possible; see details in [5]). Furthermore, we consider the network structure in the information flow graph and formulate the optimal-cost problem. Our main tool for analysis would be cut-set bound analysis in multicast networks, proposed in seminal paper [28]. In the literature, the process that a node fails and a new node is regenerated is called a stage of repair. Evolving network in a distributed storage system (infinite time leaving/joining of nodes) implies infinite number of repair stages. Cut-set bound analysis with considering infinite stages of repair in a heterogeneous network even for a distributed storage system with small number of storage nodes would be complicated. We thus try to find a lower bound of repair-cost by cut-set bound analysis in the first stages of repair. We later show that this bound can be achieved for the minimum storage regenerating codes, i.e.,  $\alpha = M/k$ . In the literature, the process that a node fails and a new node is regenerated is called a stage of repair.

#### A. Network Setup

Above, for a specific network, we have studied the repair cost and proposed SNC to reduce the cost. Naturally, we may ask what is the optimal-cost and how to achieve the optimality for more general scenarios. In this section, we first formulate a linear optimization problem which establishes a fundamental lower bound on the repair cost for general networks.

Assume in a distributed storage systems with  $n$  nodes, node  $n$  fails and  $d$  number of surviving nodes ( $k \leq d \leq n-1$ ) help to regenerate a new node. We also assume that network topology and the cost of transmitting one unit of data between nodes are given. From the network topology the paths from helper nodes to new node are known. To study the optimal-cost repair problem for a given network, assuming failure on

node  $n$ , we define an  $d \times n$  cost matrix  $\underline{C}$ , as follows,

$$\underline{C} = \begin{pmatrix} 0 & c_{(12)} & \cdots & c_{(1n)} \\ c_{(21)} & 0 & \cdots & c_{(2n)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(dn)} & \cdots & \cdots & c_{(dn)} \end{pmatrix},$$

where an element  $c_{(ij)}$  denotes the link cost of transmitting one unit fragment from node  $i$  to node  $j$ . For instance,  $c_{(12)}$  is the cost of transmitting one fragment from node 1 to node 2, and  $c_{(in)}$  is the corresponding cost from surviving node  $i$  to the new node (assuming node  $n$  fails).  $c_{(ij)}$  is nonzero if there is direct link from node  $i$  to node  $j$ . Otherwise,  $c_{(ij)} = \infty$ . We assume that an algorithm extracts the matrix  $\underline{C}$  from a given network topology and link costs. Clearly, by matrix  $\underline{C}$ , we can calculate the costs of a path (maybe multi-hop) between any pair of nodes in the network. Here, we assume there is at least one path between any pair of nodes. Furthermore, we assume that the network is delay-free and acyclic. In this paper, we only consider linear costs. That means if the transmission cost of one fragment from node  $i$  to  $j$  is  $c_{(ij)}$ , then it costs  $mc_{(ij)}$  to transmit  $m$  fragments from node  $i$  to  $j$ .

To investigate the repair-cost in a given network, we modify the information flow graph in [5] by introducing the network topology and link costs into the graph.

### B. Modified Information Flow Graph

Consider a storage system with the original file of size  $M$  distributed among  $n$  nodes in which each node stores  $\alpha$  fragments and any  $k$  of  $n$  nodes can rebuild the original file. We denote the source file with an  $M \times 1$  vector,  $\underline{s}$ . Then, the code on each node  $i$  can be represented by a matrix  $\underline{Q}_i = [\underline{q}_i^1, \dots, \underline{q}_i^\alpha]$  of size  $M \times \alpha$  where each column ( $\underline{q}_i^j$ ) represents the code coefficients of fragment  $j$  on node  $i$ , and then the stored data in node  $i$  is  $\underline{X}_i = \underline{Q}_i^T \underline{s}$ .

We can denote the flow of information in a distributed storage system by a directed acyclic graph  $G(n, k, \alpha) = G(N, A)$ , where  $N$  is the set of nodes and  $A$  is the set of directed links. Similar to [5], graph  $G(n, k, \alpha)$  consists of three different nodes: a source node, storage nodes and several data collectors (DC). The source node distributes the original file among storage nodes along with the (assumably) infinite-capacity links. Every storage node can be denoted by input ( $in$ ) and output ( $out$ ) nodes connecting by a link of capacity equals to the amount of node's storage ( $\alpha$ ). Finally, the DC reconstructs the original file by connecting to at least  $k$  storage nodes via the infinite-capacity links and then solving linear equations. Contrary to [5], there might not exist direct links from surviving nodes to the new node and storage nodes may relay other nodes' fragments to the new node. When a node fails,  $d$  surviving nodes participate in the repair process ( $k \leq d \leq n - 1$ ). Here, an optimization algorithm shall determine the optimum traffic on the links of the network. We note that in distributed storage networks, node failure and new node generation may happen infinite times. In a heterogenous network, considering all stages of repair makes the problem complicated. Hence, in our study we focus on the first stage of repair, which is a performance lower bound for infinite repair

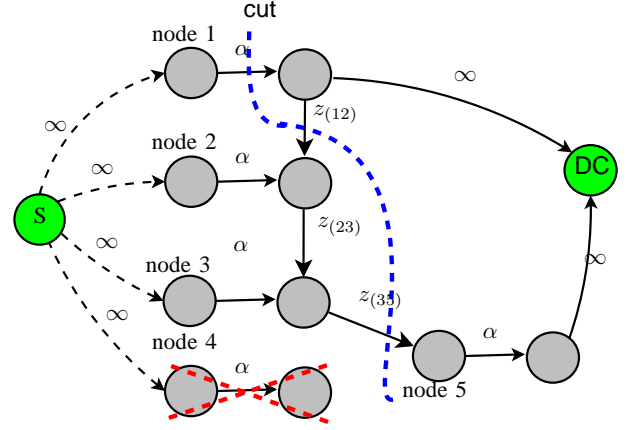


Fig. 4. Modified information flow graph for the first stage of repair in a tandem network. The cut mentioned on the figure with heavy dotted-line corresponds to the inequality  $z_{(35)} + \alpha \geq M$ .

stages. As an example, the modified information flow graph for the first stage of repair on node 4 has been shown in Fig. 4. In this example, nodes are connected in a tandem network.

We use a column vector to denote the number of fragments transmitted on the directed links. This vector is termed as a *subgraph* ( $\underline{z} = [z_{(ij)}]_{(ij) \in A}$ ) [23], where each element  $z_{(ij)}$  is a non-negative integer and represents the number of fragments from node  $i$  to node  $j$  for  $(ij) \in A$ . For the subgraph  $\underline{z} = [z_{(ij)}]_{(ij) \in A}$ , and defining  $c_{(ij)}$  as the cost of transmitting one unit of data (e.g., one fragment) from node  $i$  to node  $j$ , the repair cost can be formulated as,

$$\sigma_c(\underline{z}) \triangleq \sum_{(ij) \in A} c_{(ij)} z_{(ij)}. \quad (1)$$

Our objective is to minimize the cost ( $\sigma_c$ ) considering only the first stages of repair.

*Remark 1:* We note that in the repair problem there would be infinite number of repair stages. However, cut analysis in the first stage of repair gives the necessary conditions for finding the optimum-cost regenerating approach. The analysis gives a lower bound of repair-cost for the repair. Later, we show that this lower bound can be achieved for  $\alpha = M/k$ . Thus, in our analysis there would be limited number of cut-set constraints. Clearly, the number of cut constraints depends on the number of nodes  $n$  and  $k$ .

### C. Cut-set analysis in the first stage of repair

1) *Constraint Region:* In the repair process, it is required that any  $k$  nodes can reconstruct the original file. This property is known as the regenerating code property (RCP). The RCP must be preserved for an arbitrary number of stages of repair. For this, it is necessary for the first stage of repair all the cut constraints to be satisfied. Thus, we find the minimum  $\sigma_c$  under the constraint that in the first stage of repair, all cuts of connecting the DC to the new node and  $k - 1$  other nodes must be greater than or equal to  $M$ . In Fig. 4, the heavy dotted line presents a cut when the DC connects to the new node and node 1 for a regenerating code with parameter  $k = 2$ . The cut constraint relating to this cut can



be formulated by the inequality:  $z_{(35)} + \alpha \geq M$ . By assuming vector  $\underline{z} = [z_{(12)} z_{(23)} z_{(35)}]^T$ , we can express the inequality in vector space as,  $(0, 0, 1)\underline{z} \geq M - \alpha$ . Assume  $r$  cut constraints on the first stage of repair. Denoting  $|A|$  the cardinality of existing edges between nodes, we form all the inequalities in a matrix form, by defining an  $r \times |A|$  dimension matrix  $L$  (this matrix is called *coefficient matrix* in the literature [27]). The corresponding inequalities induced by the cut constraints show a region in a multi-dimensional space that the subgraph must satisfy to be a feasible solution. This region is often called *polytope* [27]. Consequently the polytope is

$$\Psi = \{\underline{z} = [z_{(ij)}] \mid z_{(ij)} \geq 0, \underline{L}\underline{z} \geq \underline{b}\}, \quad (2)$$

where the comparison of two vectors e.g.  $\underline{a} \geq \underline{b}$  means every element in  $\underline{a}$  is greater than or equals to the element in  $\underline{b}$  at the same position.

*Example 1:* In Fig. 4, if the DC connects to node 1 and the new node to rebuild the source ( $k = 2$ ), the first cut constraint is

$$z_{(35)} \geq M - \alpha. \quad (3)$$

The second constraint follows if we connect the DC with node 2 and the new node,

$$z_{(12)} + z_{(35)} \geq M - \alpha. \quad (4)$$

Similarly, when the DC connects to node 3 and the new node, we have the third constraint

$$z_{(23)} \geq M - \alpha. \quad (5)$$

Thus, we can form all these inequalities in a matrix form as follows,

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\underline{L}} \underbrace{\begin{bmatrix} z_{(12)} \\ z_{(23)} \\ z_{(35)} \end{bmatrix}}_{\underline{z}} \geq \underbrace{\begin{bmatrix} M - \alpha \\ M - \alpha \\ M - \alpha \end{bmatrix}}_{\underline{b}}. \quad (6)$$

*Remark 2:* We note that in the cut-set bound analysis there would be some cut constraints that do not affect the polytope region. These are called non-active constraints. In the example above we only consider the active constraints.

*Remark 3:* The polytope  $\Psi$  is restricted by linear inequalities. Hence, if  $z_{(ij)}$ s are real numbers then the constraint region  $\Psi$  is convex. We can reasonably assume that  $z_{(ij)}$ s are real numbers. Note that the file is measured by bits (integer) but it is normally quite large. Hence we can consider  $z_{(ij)}$ s as real values since one fragment has lots of bits. Following this assumption,  $\Psi$  constitutes a convex region.

2) *Linear Optimization:* Since the constraint region and objective function in the repair problem are linear, the problem is a linear optimization problem which can be solved efficiently. Finally, we can formulate the optimization problem as,

$$\begin{aligned} & \text{minimize} && \sigma_c(\underline{z}) = \sum_{(ij) \in A} c_{(ij)} z_{(ij)} \\ & \text{subject to} && \underline{L}\underline{z} \geq \underline{b}, \\ & && z_{(ij)} \geq 0. \end{aligned} \quad (7)$$

The linear programming in (7) results in a lower bound of repair costs.

*Proposition 1:* The repair-cost calculated by problem (7) is the lower bound of the repair-cost.

*Proof:* In a general network, cut-set bound analysis considering infinite stages of repair is needed to have all the constraint on  $z_{(ij)}$ s. This is the necessary and sufficient conditions on  $z_{(ij)}$  for the repair. Thus, cut-set bounds considering only the first stage of repair gives only necessary conditions, and in general they might not be sufficient conditions. Denote the polytope resulting from cut-set analysis on the first stage as  $\Psi^1$ , and the polytope given the cut-set constraints till stage  $t$  as  $\Psi^t$ , for  $t$  as a positive integer number. We have  $\Psi^t \subseteq \Psi^1$ , because for any  $\underline{z} \in \Psi^t$ , satisfying constraint till stage  $t$ ,  $\underline{z}$  have to satisfy the cut constraints at first stage, i.e.,  $\underline{z} \in \Psi^1$ . Denote  $\sigma_{opt}^t$  as the minimum value of  $\sigma_c$  satisfying all the cut constraints till stage  $t$ . In other words,  $\sigma_{opt}^t$  corresponds to the minimum  $\sigma_c$  on which plane  $\sigma_c = \sum_{(ij) \in A} c_{(ij)} z_{(ij)}$  intersects polytope  $\Psi^t$ . Since  $c_{(ij)}$ s are non-negative real numbers,  $\Psi^t \subseteq \Psi^1$ , the minimum-cost given the cut constraints till stage 1 is not greater than the minimum-cost given the cut constraints till stage  $t$ . That is  $\sigma_{opt}^1 \leq \sigma_{opt}^t$ . ■

3) *Example for the 4 node tandem network:* Now we have enough tools to find the optimum repair cost in the motivating example (Fig. 1). We assume three nodes joining the repair process ( $d = 3$ ),  $M = 4, k = 2, \alpha = 2$ , and the corresponding cost matrix is,

$$\underline{C} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty \\ \infty & \infty & 0 & 1 \end{pmatrix}. \quad (8)$$

Clearly there exists a direct link between nodes 1 and 2 with one unit cost, and similarly between nodes 2 and 3, and nodes 3 and 5 (new node). Now we analyze the constraint region for  $M = 4, \alpha = 2$  in (6). Hence, we can formulate the problem as

$$\begin{aligned} & \text{minimize} && \sigma_c(\underline{z}) = z_{(12)} + z_{(23)} + z_{(35)} \\ & \text{subject to} && \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} z_{(12)} \\ z_{(23)} \\ z_{(35)} \end{bmatrix} \geq \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}. \end{aligned} \quad (9)$$

Solving the linear optimization problem (e.g., by a simplex method [27]) gives the optimal subgraph  $(z_{(12)}, z_{(23)}, z_{(35)}) = (0, 2, 2)$  with a cost of 4 units. A linear network coding by selecting coefficients (in this example) from  $GF(5)$  with SNC in Fig. 3 can meet the minimum-cost subgraph. The coding scheme is  $\underline{p}_1 = 0, \underline{p}_2 = 2\underline{a}_2 + \underline{b}_2, \underline{p}_3 = \underline{a}_2 + 2\underline{b}_2, \underline{p}_4 = \underline{p}_2 + (\underline{a}_1 + \underline{b}_1 + \underline{a}_2 + \underline{b}_2) = \underline{a}_1 + \underline{b}_1 + 3\underline{a}_2 + 2\underline{b}_2$  and  $\underline{p}_5 = \underline{p}_3 + (\underline{a}_1 + 2\underline{b}_1 + \underline{a}_2 + 2\underline{b}_2) = \underline{a}_1 + 2\underline{b}_1 + 2\underline{a}_2 + 4\underline{b}_2$ . Here  $\underline{p}_4, \underline{p}_5$  are fragments for the new node, which satisfies RCP.

#### IV. OPTIMAL-COST CODE FOR THE MINIMUM STORAGE NETWORK ( $\alpha = M/k$ )

In this section, we show that the lower bound of the repair-cost is achievable for  $\alpha = \frac{M}{k}$ . That is, there exists a linear network code corresponding to the repair with the minimum-cost subgraph from the optimization problem (7). We call the codes that achieve this optimal point as the optimal-cost minimum-storage regenerating (OCMSR) code. Our proof is based on random linear codes and then we discuss the required

finite field size for constructing the OCMSR code. Similar to the method in [6], we consider the first stage of repair and then by induction on the number of repair stages, we generalize the results to multiple stages of repairs. To find the sufficient field size for successful regeneration, we apply sparse-zero lemma as follows.

*Lemma 1:* Consider a multi-variable polynomial  $g(\alpha_1, \alpha_2, \dots, \alpha_n)$  which is not identically zero, and has the maximum degree in each variable at most  $d_0$ . Then, there exist variables  $\gamma_1, \gamma_2, \dots, \gamma_n$  in the finite field  $GF(q)$ , and  $q \geq d_0$ , such that  $g(\gamma_1, \gamma_2, \dots, \gamma_n) \neq 0$

*Proof:* See proof of Lemma 19.17 in [28]. ■

Suppose a source information file consists of  $M = k\alpha$  fragments. Assuming each fragment  $\underline{m}_i$ , ( $i = 1, \dots, M$ ) being a vector of elements in  $\mathbb{F}_q$  we can denote the source by a vector  $\underline{s} = [\underline{m}_1, \dots, \underline{m}_M]^T$ . Then vector  $\underline{s}$  is encoded by an erasure code (satisfying the RCP) to  $n\alpha$  fragments and distributed among  $n$  nodes such that every storage node stores  $\alpha$  fragments. If  $\underline{X}_i$  denotes the stored symbols of node  $i$ , then  $\underline{X}_i = \underline{Q}_i^T \underline{s}$ , where  $\underline{Q}_i$  is an  $M \times \alpha$ -dimensional matrix representing the coding coefficients of node  $i$ . When a node fails (without loss of generality, we assume node 1 using the encoding coefficients  $\underline{Q}_1$  fails) the optimization algorithm finds the minimum-cost subgraph. Following the minimum-cost subgraph, the new node is regenerated by surviving node cooperation. Clearly, with the minimum-cost subgraph, we also know which nodes should encode on the directed graph. Then, with a proper finite field, we can find the network code and regenerate the new node with the coding coefficients  $\underline{Q}'_1$ , and  $\underline{X}'_i = \underline{Q}'_1{}^T \underline{s}$ .

To maintain the RCP after the regeneration of the lost node, the coding coefficients ( $\underline{Q}'_1$ ) have to meet certain requirement. That is, for any selection of  $k - 1$  out of  $n - 1$  surviving nodes,  $\Xi_{k-1} = \{\underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}\}$ , together with the codes of the new node  $\underline{Q}'_1$ , the polynomial  $\det([\underline{Q}'_1, \underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}])$  is a non-zero polynomial. In what follows, we first show that  $\det([\underline{Q}'_1, \underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}])$  satisfying the subgraph of the optimization process is not identically zero and then discuss the required field size.

*Lemma 2:* For regenerating node 1, there exist linear codes satisfying the minimum-cost subgraph (resulted from problem (7)) such that the polynomial  $\det([\underline{Q}'_1, \underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}])$  is non-zero for any selected set  $\Xi_{k-1}$ . That is,

$$\prod_{\{s_1, \dots, s_{k-1}\} \subset \{2, \dots, n\}} \det([\underline{Q}'_1, \underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}]) \neq 0. \quad (10)$$

*Proof:* Consider  $\Xi_k = \{\underline{Q}_{s_1}, \dots, \underline{Q}_{s_k}\}$  a set of coding coefficients selected from  $k$  out of  $n$  nodes. Since every  $k$  nodes can reconstruct the original file, then the matrix  $[\underline{Q}_{s_1}, \dots, \underline{Q}_{s_k}]$  has full rank  $M = k\alpha$ . Thus, for  $\Xi_{k-1}$ , the matrix  $[\underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}]$  has rank  $(k - 1)\alpha$ . To regenerate a new node with the RCP, the minimum-cost subgraph of the information flow graph meets the requirement that by connecting the *DC* to the new node and any  $k - 1$  other nodes, all the cuts are greater than (or equivalent to)  $M$ . This requires the matrix of coding coefficients of the cut in the modified flow graph containing the new node and selection

node set having full rank  $M$ . To prove this, consider a set  $\mathcal{V}$  containing the data collector, *in* and *out* nodes of the new node, and *out* nodes of the nodes in set  $\Xi_{k-1}$ . Other nodes including the source node are in the complement set  $\bar{\mathcal{V}}$ . The cut passes the *in-out* edges of nodes in  $\Xi_{k-1}$  has the rank  $(k - 1)\alpha$ . Since all the cuts has at least  $M$  edges, there would be at least  $R = M - (k - 1)\alpha = \alpha$  edges from  $\bar{\mathcal{V}}$  entering  $\mathcal{V}$ . In  $\bar{\mathcal{V}}$ , there exist  $\alpha$  vectors e.g., vectors in  $\underline{Q}_{s_k}$  which are independent of vectors in  $\Xi_{k-1}$ . Thus, if we send fragments corresponding to those  $\alpha$  independent vectors through  $R = \alpha$  edges to the  $\mathcal{V}$ , the matrix of the coding coefficients of the cut will be full rank. Therefore,  $\det([\underline{Q}'_1, \underline{Q}_{s_1}, \dots, \underline{Q}_{s_{k-1}}])$  can be non-zero. ■

To find the required field size of OCMSR codes, we need to know the maximum degree of the variables of the polynomial in (10). When SNC is allowed in intermediate nodes, as in our repairing scheme, the polynomial degree can be greater than non-SNC schemes since SNC may involve extra network coding processes, compared to the repairing schemes only encoding in the storage nodes and the new node. For analysis, we use  $n_{nc}$  to denote the maximum number of network coding processes involved by one fragment in one repair stage. We note that  $n_{nc} \leq |N|$ , the number of nodes in the networks.

*Example 1:* In the repair process of node four in tandem network (Fig. 4),  $n_{nc} = 4$ . The network coding is at node 1, 2, 3 and at the new node.

*Theorem 1:* For a distributed storage system  $G(n, k, \alpha)$  with the source file of size  $M$ , if the field size is greater than  $d_0$ , there exists a linear network code such that the RCP is satisfied for any stage of repair, regardless of how many failures/repairs happened before, where

$$d_0 = \binom{n}{k} M n_{nc}. \quad (11)$$

*Proof:* The proof is by induction on the number of repair stages. That is, we assume before a node fails all the storage nodes have the RCP. In each stage of repair, when a node fails, the new node is regenerated preserving the RCP. Thus, we initialize the code on  $n$  nodes by which any  $k$  out of  $n$  nodes can reconstruct the original file. Then if a node fails, the new node is regenerated such that the repairing cost is minimized and the RCP is preserved. By the RCP, the coding coefficients of any  $k$  nodes must have full rank  $M$ . That is,

$$\prod_{\{s_1, \dots, s_k\} \subset \{1, \dots, n\}} \det([\underline{Q}_{s_1}, \dots, \underline{Q}_{s_k}]) \neq 0. \quad (12)$$

Clearly, the maximum degree of variables in (12) is  $\binom{n}{k} M$ . Thus, by Lemma 1, if the field size ( $q$ ) is greater than  $\binom{n}{k} M$ , then there is a network coding solution for repair. Since  $n_{nc} \geq 2$  (at least two coding process: one in surviving nodes, and another in the new node),  $d_0 \geq \binom{n}{k} M$ ; thus, there is a coding solution for  $q \geq d_0$ .

When a node fails (assume  $Q_1$ ), the optimization algorithm finds the minimum-cost subgraph. Accordingly, the fragments are combined using linear network coding, and then the new node is regenerated. The set including the new node ( $Q'_1$ ) and

surviving nodes must satisfy the RCP. Thus,

$$\prod_{\{s_1, \dots, s_{k-1}\} \subset \{2, \dots, n\}} \det([Q'_1, Q_{s_1}, \dots, Q_{s_{k-1}}]) \neq 0. \quad (13)$$

By Lemma 2, the polynomial can be nonzero. The maximum degree of each variable is less than  $\binom{n-1}{k-1} M n_{nc}$ . By Lemma 1, if the finite field size  $q \geq \binom{n-1}{k-1} M n_{nc}$ , there is a network solution for the repair. Clearly,  $d_0 = \binom{n}{k} M n_{nc} \geq \binom{n-1}{k-1} M n_{nc}$  for  $n \geq k$ . Hence, for  $q > d_0$ , there exist a code for the repair. This concludes our proof. ■

In summary, optimal-cost repair for the minimum storage regenerating (OCMSR) code is given in two steps. First, the optimal-cost subgraph is found decoupled from coding by solving problem (7). Then, we can construct the code of the new node by random linear coding [25] or deterministic [26] from the finite field size determined by Theorem 1.

## V. REPAIR-COST LOWER BOUND, AND THE GAIN OF OPTIMAL-COST REPAIR IN NETWORKS WITH GIVEN TOPOLOGIES

In this section, we study the lower bound of the repair-cost for distributed storage systems on networks with given topologies. We aim at showing how considering networks and cost together result in a lower cost in the repair. We first apply our method to tandem, star and grid networks, where there might not exist direct links between new nodes to all the surviving nodes. Next, for the scenario that the new node has direct links to surviving nodes (as [5], [6]), we show that surviving node cooperation can reduce the cost. For the purpose, we study the repair cost in a fully connected network. It may worth to note that our approach can be applied on any network, and for any cost on the links of the networks. The networks considered in this section are just examples to present the gain of optimal-cost repair. For simplicity, we assume in this section that links have unit cost of transmission, unless otherwise stated. We define the gain in our approach as the ratio of repair-cost in minimum-bandwidth approach (denoted as  $\sigma_{non-opt}$ ) to the optimal-cost repair (denoted as  $\sigma_{opt}$ ),

$$g_c = \frac{\sigma_{non-opt}}{\sigma_{opt}}. \quad (14)$$

### A. Tandem Networks

1) *Repair-Cost Lower Bound in Tandem Networks:* In a tandem topology, nodes are in a line. That is, each node is linked to two neighboring nodes, except the nodes in the line ends, which have only one neighbor. When a node fails and a new node joins, the repair traffic is relayed by intermediate nodes to the new node. Then we can formulate the repair-cost lower bound by the following proposition.

*Proposition 2:* Consider a tandem distributed storage network consisting of  $n$  nodes where each node stores  $\alpha$  fragments and every  $k$  nodes can reconstruct the original file of size  $M$ . Assuming the cost of a link (between adjacent nodes) equals to one unit, the lowest repair-cost is achieved by cooperation of the  $k$  nearest surviving nodes and equals to

$$\sigma_c \geq [k(M - (k-1)\alpha)]^+, \quad (15)$$

where  $[x]^+ = \max\{x, 0\}$

*Proof:* See appendix A.

By Proposition 2, we can calculate the gain of optimized cost to the non-optimized cost approach. Gain in the repair for one of the end nodes in line can be calculated as follows.

*Corollary 1:* Consider a distributed storage system with parameters  $(M = k(n-k), \alpha = (n-k), d = n-1)$  in a tandem network with  $n$  nodes in order (node 1-node 2-...-node  $n$ ). If node 1 fails, the surviving node cooperation gives the gain

$$g_c = \frac{n(n+1)}{2k(n-k)}. \quad (16)$$

*Proof:* With the given parameters the minimum bandwidth repair is  $\beta = 1$  [5]. Sending the repair bandwidth from node  $n$  to the new node costs  $n$  units, from node  $n-1$  to the new node costs  $n-1$  units and so on. Thus in total the repair cost in this approach is  $\sigma_{non-opt} = n + (n-1) + \dots + 1 = \frac{n(n+1)}{2}$ . From Proposition 2,  $\sigma_c = k(n-k)$ . Hence the gain would be  $g_c = \frac{n(n+1)}{2k(n-k)}$ .

Therefore, for  $(n, k)$ -MDS codes with  $k = n/2$  the  $g_c$  tends to 2 for a large network ( $n \rightarrow \infty$ ).

2) *Explicit Construction of OCMSR code for Tandem Networks:* Previously, we proved that the optimal cost functional repair can be achieved by applying linear network coding in a large finite field. In this section, we show that the lower bound can be achieved even with an extra condition of exact repair. We give an explicit construction for the optimal-cost exact repair in tandem networks. The motivation is that it requires smaller finite field size comparing to random code and also it has explicit construction. We split the source file of size  $M$  into  $k$  fragments. We denote the source file by vector  $\underline{m} = [m_1 m_2 \dots m_k]^T$ . To construct  $(n, k)$ -MDS code, We use a  $k \times n$  Vandermonde matrix  $G$  as a generator matrix.

$$G = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_n^{k-1} \end{pmatrix}, \quad (17)$$

where  $\alpha_i$ s for  $i \in \{1, \dots, n\}$  are distinct elements from the finite field  $GF(q)$ .

By the property of Vandermonde matrix, every  $k \times k$  submatrix of  $G$  is full rank if  $\alpha_i$ s, for  $i \in \{1, \dots, n\}$ , are distinct elements. This requires  $q \geq n$ . Each column i.e., column  $t, t \in 1, \dots, n$  in matrix  $G$  represents the code on node  $t$ . We denote the coded data on node  $t$  as  $v(t)$ , then,

$$v_t = m_1 + m_2 \alpha_t + \dots + m_k \alpha_t^{k-1} = [1 \ \alpha_t \ \dots \ \alpha_t^{k-1}] \underline{m}. \quad (18)$$

By the property of Vandermonde matrix, every  $k \times k$  submatrix of  $G$  is full rank then a data collector can reconstruct the source file by connecting to any  $k$  nodes. The regenerating process is only by linear combination. Assume nodes are labeled in order i.e., node 1 connects to node 2, node 2 connects to node 1 and 3, and so on. By Proposition 2, for  $M = k$  fragments, the optimum-cost repair is by transmitting  $M/k = k/k = 1$  fragment to the neighbor. Assume node  $t$ ,



for  $t \in \{1, \dots, n\}$ , fails and a sequence of backward nodes  $\{\text{node}_{t-k_1}, \text{node}_{t-k_1+1}, \dots, \text{node}_{t-1}\}$  and a sequence of forward nodes  $\{\text{node}_{t+1}, \text{node}_{t+2}, \dots, \text{node}_{t+k_2}\}$ , such that  $k_1 + k_2 = k$ , help the new node to regenerate the content of the failed node.

The repair process is as follows. The new node receives fragments from two directions, from node  $t-1$ , aggregating data from nodes  $\text{node}_{t-k_1}$  till node  $t-1$ , and from node  $t+1$ , aggregating data from nodes  $\text{node}_{t+k_2}$  to node  $t+1$ . Thus, in one direction node  $\text{node}_{t-k_1}$  multiplies its content by a coefficient  $\xi_{t-k_1}$  from  $GF(q)$  and sends the result to node  $\text{node}_{t-k_1+1}$ . Then node  $\text{node}_{t-k_1+1}$ , multiplies its content by  $\xi_{t-k_1+1}$  and combines the result to the received fragment and then sends its combined fragment to its next neighbor  $\text{node}_{t-k_1+2}$ . Finally node  $t-1$  transmits the combined fragment  $w_{t-1}$ , which is,

$$\begin{aligned} w_{t-1} &= \xi_{t-k_1} v_{t-k_1} + \xi_{t-k_1+1} v_{t-k_1+1} + \dots + \xi_{t-1} v_{t-1} \\ &= \xi_{t-k_1} (m_1 + m_2 \alpha_{t-k_1} + \dots + m_k \alpha_{t-k_1}^{k-1}) \\ &\quad + \xi_{t-k_1+1} (m_1 + m_2 \alpha_{t-k_1+1} + \dots + m_k \alpha_{t-k_1+1}^{k-1}) \\ &\quad + \dots + \xi_{t-1} (m_1 + m_2 \alpha_{t-1} + \dots + m_k \alpha_{t-1}^{k-1}). \end{aligned} \quad (19)$$

In another route, node  $\text{node}_{t+1}$  sends the aggregated fragment to the new node. That is,  $\text{node}_{t+k_2}$  multiplies its content by a coefficient  $\xi_{t+k_2}$  from  $GF(q)$  and sends the result to  $\text{node}_{t+k_2-1}$ . Then node  $\text{node}_{t+k_2-1}$ , multiplies its content by  $\xi_{t+k_2-1}$  and combines the result to the received fragment and then sends its combined fragment to its next neighbor  $\text{node}_{t+k_2-2}$ . Other helping nodes do the same. Finally node  $t+1$  transmits the combined fragment  $w_{t+1}$ , which is,

$$\begin{aligned} w_{t+1} &= \xi_{t+k_2} v_{t+k_2} + \xi_{t+k_2-1} v_{t+k_2-1} + \dots + \xi_{t+1} v_{t+1} \\ &= \xi_{t+k_2} (m_1 + m_2 \alpha_{t+k_2} + \dots + m_k \alpha_{t+k_2}^{k-1}) \\ &\quad + \xi_{t+k_2-1} (m_1 + m_2 \alpha_{t+k_2-1} + \dots + m_k \alpha_{t+k_2-1}^{k-1}) \\ &\quad + \dots + \xi_{t+1} (m_1 + m_2 \alpha_{t+1} + \dots + m_k \alpha_{t+1}^{k-1}). \end{aligned} \quad (20)$$

In order to achieve exact repair, we put  $w_{t-1} + w_{t+1} = v_t$

$$v_t = m_1 + m_2 \alpha_t + \dots + m_k \alpha_t^{k-1}. \quad (21)$$

Thus, vector  $\underline{\xi} = [\xi_{t-k_1}, \dots, \xi_{t-1}, \xi_{t+1}, \dots, \xi_{t+k_2}]^T$  should be selected such that,

$$\begin{aligned} &[\xi_{t-k_1} \xi_{t-k_1+1} \dots, \xi_{t+k_2}] \times \\ &\underbrace{\begin{pmatrix} 1 & \alpha_{t-k_1} & \alpha_{t-k_1}^2 & \dots & \alpha_{t-k_1}^{k-1} \\ 1 & \alpha_{t-k_1+1} & \alpha_{t-k_1+1}^2 & \dots & \alpha_{t-k_1+1}^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{t+k_2} & \alpha_{t+k_2}^2 & \dots & \alpha_{t+k_2}^{k-1} \end{pmatrix}}_{\underline{A}} = \begin{pmatrix} 1 \\ \alpha_t \\ \alpha_t^2 \\ \vdots \\ \alpha_t^{k-1} \end{pmatrix}^T \end{aligned} \quad (22)$$

Since matrix  $\underline{A}$  is non-singular, we can always find linear codes ( $\underline{\xi}$ ) that make the exact repair possible. Hence, for the successful reconstruction and repair process the finite field  $q > n$  suffices.

### B. Star Networks

In a star network, there is a central node, and any pair of non-central nodes must communicate through the central node.

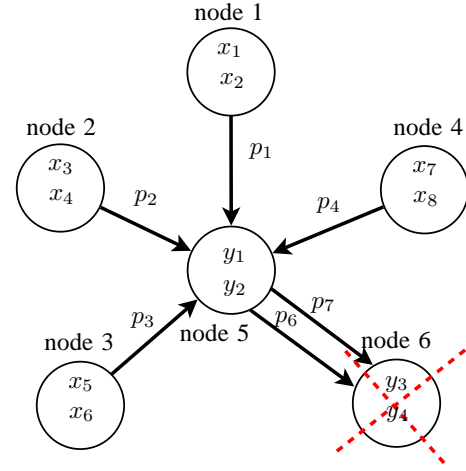


Fig. 5. Optimal-cost repair in a star network with 6 nodes.

Fig. 5 shows an example of a distributed storage system in a star network. In the scenario, if the central node fails, and all link costs are equivalent, the optimal-cost repair reduces to the optimal-bandwidth repair in [5]. However, if non-central nodes fail, the situation is different. Assuming  $n$  storage nodes and a source file size  $M$ , the lower bound of repair-cost for a non-central node is calculated as follows.

**Proposition 3:** In a distributed storage system with a star topology, the repair-cost of a non-central is greater than or equals to

$$\sigma_c \geq \left(\frac{n-2}{n-k} + 1\right)[M - (k-1)\alpha]^+. \quad (23)$$

*Proof:* See appendix B.

By Proposition 3, we can calculate the gain of surviving node cooperation comparing to the non-optimized cost approach for non-central nodes, as follows.

**Corollary 2:** Consider a distributed storage system with parameters  $(M = k(n-k), \alpha = (n-k), d = n-1)$  in a star network with  $n$  nodes. If a non-central node fails, the surviving node cooperation gives the gain  $g_c$ ,

$$g_c = \frac{2n-3}{2n-k-2}. \quad (24)$$

*Proof:* With the given parameters the minimum bandwidth repair is  $\beta = 1$  [5]. Sending the repair traffic from  $(n-2)$  non-central nodes to the central node costs  $(n-2)$ . The central node sends these  $n-2$  fragments plus its own repair. This means the central node sends  $(n-1)$  fragments to the new node. In total the repair cost equals to  $\sigma_{non-opt} = (n-2) + (n-1) = 2n-3$ . For the optimal cost, we calculate from Proposition 3,  $\sigma_c = \left(\frac{n-2}{n-k} + 1\right) \frac{M}{k}$ . Substituting  $M = k(n-k)$  yields  $\sigma_c = 2n-k-2$ .

Therefore, as an example for  $k = n/2$ ,  $g_c$  tends to  $\frac{4}{3}$  for  $n \rightarrow \infty$ .

### C. Grid Networks

Consider a  $2 \times 3$  grid network in Fig. 6. The optimal-cost repair process depends on the location of the new node. The

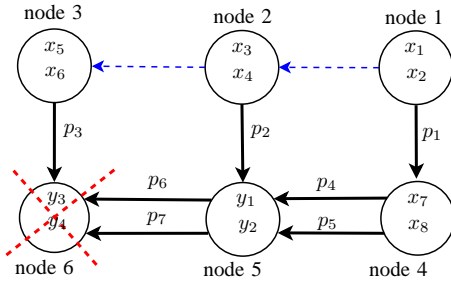


Fig. 6. Optimization repair in the  $2 \times 3$  grid network. Dashed lines show available links which are not used in the repair process.

optimum cost repair for the repair in node 6 can be found by our approach.

*Claim 1:* In the repair of node 6 in Fig. 6, the optimal-cost repair is 7 units corresponding to the minimum-cost subgraph  $(z_{(12)}, z_{(14)}, z_{(23)}, z_{(25)}, z_{(36)}, z_{(45)}, z_{(56)}) = (0, 1, 0, 1, 1, 2, 2)$ .

*Proof:* See appendix C.

In this example in the non-optimized cost approach, the minimum bandwidth for  $M = 8, n = 6, k = 4, d = n - 1$  is  $\beta = 1$  fragment [5]. Surviving nodes transmit their fragment to the new node following this approach: node 3 by the path  $\text{node1} \rightarrow \text{node2} \rightarrow \text{node3} \rightarrow 6$ , node 2 by  $\text{node2} \rightarrow \text{node3} \rightarrow 6$ , node 3 by  $\text{node3} \rightarrow 6$ , node 4 by  $\text{node4} \rightarrow \text{node5} \rightarrow 6$ , and finally node 5 by transmitting a fragment on link  $\text{node5} \rightarrow 6$ . This gives  $\sigma_{\text{non-opt}} = 3 + 2 + 1 + 2 + 1 = 9$  units. Thus, our approach gives a gain  $g_c = \frac{9}{7} = 1.2$ .

Finding the lower bound of the repair cost in a closed form for a grid network is more complicated than those for tandem and star networks. Yet, we can know the repair-cost in a grid network will not be greater than the repair-cost in the tandem topology if other conditions are the same, e.g., link costs, the number of nodes. More formally, we have

*Corollary 3:* Optimal-cost repair for a distributed storage system in a grid network with  $n$  node coded by  $(n, k)$ -MDS codes (each node stores  $M/k$  fragments) leads less repair-cost than that of a tandem topology if all other conditions are the same. That is,

$$\sigma_c^{\text{grid}} \leq \sigma_c^{\text{tandem}} = [k(M - (k - 1)\alpha)]^+. \quad (25)$$

*Proof:* The repair process in a grid network can be reduced to that of tandem networks if we neglect some available links which may reduce the cost. Thus, the cost of optimal-cost repair of a grid network is upper-bounded by that of a tandem topology. ■

#### D. Fully connected network

In a fully connected network all nodes have direct links to each other. This model suits for data centers where each data center is connected to another one through a hierarchical network structure [17]. For an example, consider a network with 5 nodes where every pair of nodes are connected by direct links. We shall show that there is a gain in SNC also in fully connected networks. For this, first assume the transmission of

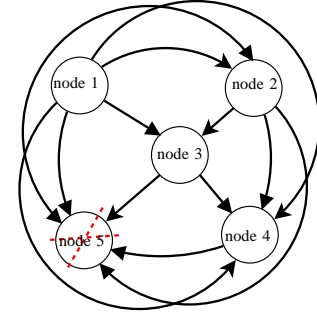


Fig. 7. Optimal-cost repair in a fully connected network with 5 nodes.

one unit of data on all links costs one unit. In this case, the optimal cost approach is to transmit from surviving nodes to the new node (without surviving node cooperation) and the cost is equivalent to the approach in [5].

*Proposition 4:* For a distributed storage in Fig.7, with parameters  $(M = 6, n = 5, k = 3, d = 4, \alpha = M/k)$  in a fully connected network with equal transmission cost on links, the optimum-cost approach is one-hop transmission, namely, directly from surviving nodes to the new node.

*Proof:* See appendix D.

Proposition 4 holds for a fully connected network with all the links having equivalent costs. In the case that there are different costs, then it might not be optimum-cost to follow the same approach (direct transmission). In this case it might be better to exploit the multi-hop network structure. For example, consider a scenario where links from surviving nodes to the new node have high cost, however links between surviving nodes are low in cost (e.g., some nodes which are located close together in one region want to send data to the new node located very far from them). In this scenario surviving nodes can cooperate and only send the aggregated data of size  $\alpha$  to the new node. More formally, we have the following result.

*Proposition 5:* Consider a distributed storage in Fig.7 having same parameters as Proposition 4. The only difference is that transmitting one fragment from surviving nodes to the new node costs 3 units, i.e.,  $c_{(15)} = c_{(25)} = c_{(35)} = c_{(45)} = 3$ . Then the optimal repair cost equals to 10 and the optimal-cost repair is through links  $z_{(23)} = z_{(34)} = z_{(45)} = 2$ .

*Proof:* See appendix E.

Thus,  $\sigma_{\text{opt}} = 2 \times 1 + 2 \times 1 + 2 \times 3 = 10$ . Yet, the repair by direct transmission costs 12 units, which is suboptimal. Hence, the optimal-cost approach, using surviving node cooperation, gains  $\frac{12}{10}$  in reducing the repair-cost.

## VI. CONCLUSIONS

We have studied repair costs for distributed storage systems with network coding in heterogeneous networks. We formulated a linear programming problem which gives the fundamental lower bound of the repair cost. The linear programming also leads us to the optimal cost approach for the minimum storage per node, (OCMSR) codes. We prove the existence of the code by the random linear code in a large finite field. We discussed the required field size for the existence of the code. To reduce the cost in networks, we proposed surviving node

cooperation. We also proposed an explicit construction for the exact optimal-cost repair in a tandem network. For specific networks with tandem, grid and star topology, we give the closed-form gains of optimal repair. We also discuss the gain of our approach for networks with fully connected topology.

## VII. APPENDICES

### A. Proof of Proposition 2

Intuitively, every  $k$  nodes can reconstruct the original file. Thus, if  $k$  nodes are allowed to combine their fragments (possibly using SNC) then they can regenerate the new node. When the cost of transmission on each channel is the same, the  $k$  closest neighbors to the new node use the least transmission costs on the network. Thus, they may be the optimal-cost solution. A stricter proof is as follows.

Assume  $n$  nodes are labeled from 1 to  $n$  as shown in Fig. 8. Without loss of generality, we assume node 1 fails and the new node ( $1'$ ) is regenerated. We denote the constraint region in a general form and then find the minimum repair cost value. We obtain the constraint region by the following steps:

Step 1: connect the DC to the new node and nodes  $n, \dots, n-k+1$ , as shown in Fig. 8. Then the cut constraint is,

$$z_{(21')} + (k-1)\alpha \geq M \Rightarrow z_{(21')} \geq M - (k-1)\alpha. \quad (26)$$

Step 2: connect the DC to the new node, node 2 and nodes  $n, \dots, n-k+2$ . Then the cut constraint is,

$$z_{(32)} + (k-1)\alpha \geq M \Rightarrow z_{(32)} \geq M - (k-1)\alpha. \quad (27)$$

Step 3: connect the DC to the new node, node 2,  $\dots, j-1$  and nodes  $n, \dots, n-k+j-1$  for  $j = 3, \dots, k+1$ . Then the cut constraint is,

$$z_{(jj-1)} + (k-1)\alpha \geq M \Rightarrow z_{(jj-1)} \geq M - (k-1)\alpha, \quad (28)$$

Other constraints of connecting DC to the new node and  $k-1$  surviving nodes are non-active constraints<sup>1</sup> in the problem. Finally, the active constraints are

$$\begin{cases} z_{(21')} \geq M - (k-1)\alpha, \\ z_{(32)} \geq M - (k-1)\alpha, \\ \vdots \\ z_{(k+1k)} \geq M - (k-1)\alpha, \\ z_{(k+2k+1)} \geq 0, \\ \vdots \\ z_{(n-1n-2)} \geq 0, \\ z_{(nn-1)} \geq 0. \end{cases} \quad (29)$$

Therefore, summing both sides of inequalities yields,

$$\sigma_c = z_{(21')} + z_{(32)} + \dots + z_{(nn-1)} \geq k(M - (k-1)\alpha). \quad (30)$$

<sup>1</sup>A non-active constraint is a constraint that does not affect or change the constraint region.

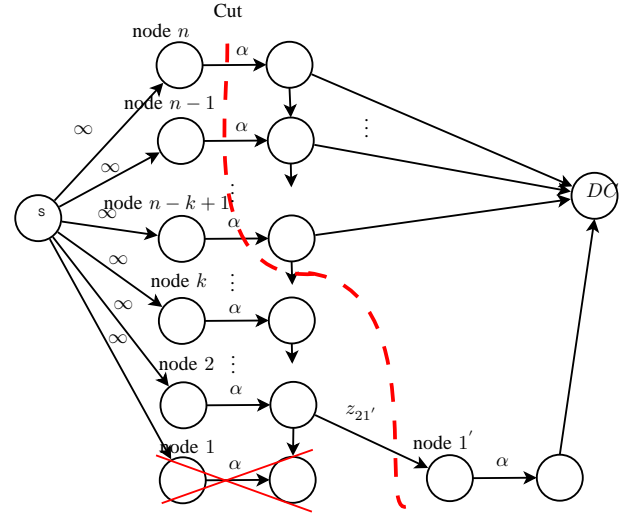


Fig. 8. Cut in tandem network. This figure corresponds to the cut of  $z_{(21')} + (k-1)\alpha \geq M$ .

### B. Proof of Proposition 3

We draw the information flow graph, as shown in Fig. 9, and prove the results by the following steps:

- Step 1: connect the DC to the new node, and nodes  $n, \dots, n-k+1$ , as shown in Fig. 9. Then the cut constraint is,

$$z_{(21')} + (k-1)\alpha \geq M, \Rightarrow z_{(21')} \geq M - (k-1)\alpha. \quad (31)$$

- Step 2: connect the DC to the new node, and node 2 and  $(k-2)$  other nodes. We see for all  $\binom{n-2}{k-2}$  selections of other nodes, each  $z_{(ij)}$  appears  $\binom{n-3}{k-2}$ . Thus if we add both sides of the constraints, we have

$$\begin{aligned} \binom{n-3}{k-2}(z_{n2} + \dots + z_{32}) + \binom{n-2}{k-2}(k-1)\alpha &\geq \binom{n-2}{k-2}M, \\ \Rightarrow \sigma_c - z_{(21')} &\geq \frac{\binom{n-2}{k-2}}{\binom{n-3}{k-2}}(M - (k-1)\alpha). \end{aligned} \quad (32)$$

- Step 3: Any other cut constraint is non-active constraint. therefore,

$$\sigma_c \geq \left(\frac{\binom{n-2}{k-2}}{\binom{n-3}{k-2}} + 1\right)(M - (k-1)\alpha) = \left(\frac{n-2}{n-k} + 1\right)(M - (k-1)\alpha). \quad (34)$$

### C. Proof of Claim 1

The corresponding cost matrix ( $\underline{C}$ ) for the repair on node 6 is as below. We formulate the linear optimization problem as follows. There are  $\binom{6-1}{4-1} = 10$  active cut constraints. Fig. 10 shows one of these cut constraints.

$$\underline{C} = \begin{pmatrix} 0 & 1 & \infty & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty & 1 & \infty \\ \infty & \infty & 0 & \infty & \infty & 1 \\ \infty & \infty & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & 0 & 1 \end{pmatrix}. \quad (35)$$

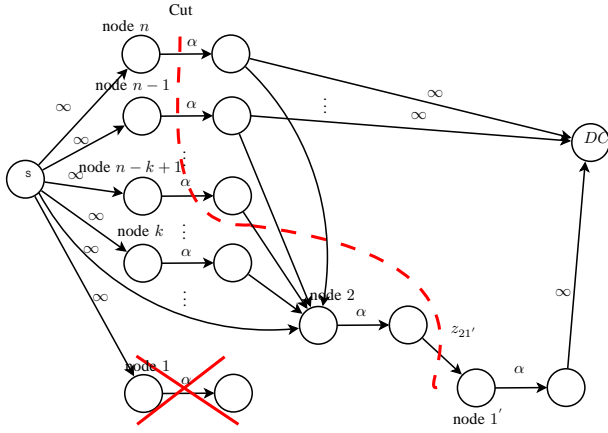


Fig. 9. Cut analysis in star network. node 2 is central node. This figure corresponds to the cut constraint of  $z_{(21')} + (k-1)\alpha \geq M$ .

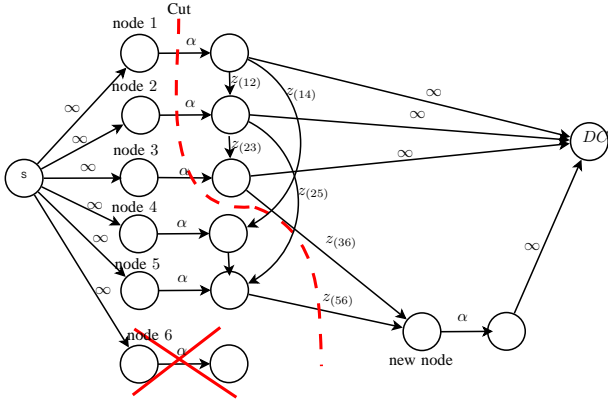


Fig. 10. Cut analysis in the  $2 \times 3$  grid network with  $n = 6, k = 4$ . The cut in this figure corresponds to the inequality,  $z_{(56)} \geq M - 3\alpha$

$$\begin{aligned} \min \quad & \sigma_c(\underline{z}) \\ \text{s.t.} \quad & \underline{H} \cdot \underline{z} \geq (M - 3\alpha)\underline{\mathbf{1}} \\ & z_{ij} \geq 0, \end{aligned} \quad (36)$$

where  $\underline{H} =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \underline{z} = \begin{pmatrix} z_{12} \\ z_{14} \\ z_{23} \\ z_{25} \\ z_{36} \\ z_{45} \\ z_{56} \end{pmatrix},$$

where  $\underline{\mathbf{1}}$  is a  $10 \times 1$  vector with all the entities equal to one). We note that matrix  $H$  is resulted from cut-set analysis in the first stage of repair. For example the cut mentioned in Fig. 10 constructs the 8-th row in matrix  $H$ , corresponding to inequality  $z_{(56)} \geq M - 3\alpha$ .

Solving this linear optimization problem (e.g. by the simplex method) for the  $M = 8$  and  $\alpha = 2$  results:  $\sigma_c = 7$ ;

$$(z_{(12)}, z_{(14)}, z_{(23)}, z_{(25)}, z_{(36)}, z_{(45)}, z_{(56)}) = (0, 1, 0, 1, 1, 2, 2). \blacksquare$$

#### D. Proof of Proposition 4

The cost matrix corresponding to regenerate the new node on node 5 equals to,

$$\underline{C} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ \infty & 0 & 1 & 1 & 1 \\ \infty & \infty & 0 & 1 & 1 \\ \infty & \infty & \infty & 0 & 1 \end{pmatrix}. \quad (37)$$

The cut analysis in first stage of repair give the following linear programming problem,

$$\begin{aligned} \min \quad & \sigma_c(\underline{z}) = \underline{\mathbf{1}}^T \underline{z} \\ \text{s.t.} \quad & \underline{H} \cdot \underline{z} \geq (M - 2\alpha)\underline{\mathbf{1}}, \\ & z_{(ij)} \geq 0, \end{aligned} \quad (38)$$

where  $\underline{\mathbf{1}}$  is a column vector with 10 elements all equal to 1, and  $T$  indicates vector transpose, and vector  $\underline{z} = (z_{(12)}, z_{(13)}, z_{(14)}, z_{(15)}, z_{(23)}, z_{(24)}, z_{(25)}, z_{(34)}, z_{(35)}, z_{(45)})$ , and  $\underline{H}$  calculated from cut set analysis equals to:

$$\underline{H} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Solving the linear programming problem for  $M = 6, \alpha = 2$  gives the minimum repair cost as,  $z_{(15)} = z_{(25)} = z_{(35)} = z_{(45)} = 1$  and zero for other links. This shows in this example the minimizing repair cost approach gives the same minimum value as minimizing repair bandwidth.

#### E. Proof of Proposition 5

The cut-analysis is same as Proposition 4. The only difference is in the cost function. Thus, the optimization problem would be,

$$\begin{aligned} \min \quad & \sigma_c(\underline{z}) = (1, 1, 1, 3, 1, 1, 3, 1, 3, 3)\underline{z} \\ \text{s.t.} \quad & \underline{H} \cdot \underline{z} \geq (M - 2\alpha)\underline{\mathbf{1}}, \\ & z_{(ij)} \geq 0, \end{aligned} \quad (39)$$

where  $\underline{H}, M, \alpha$  are same as Proposition 4. Solving the linear programming problem gives the optimum solution as  $z_{(23)} = z_{(34)} = z_{(45)} = 2$  and  $\sigma_c = (1, 1, 1, 3, 1, 1, 3, 1, 3, 3)\underline{z} = z_{(23)} \times 1 + z_{(34)} \times 1 + z_{(45)} \times 3 = 10$ .

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